Description of compatible differential-geometric Poisson brackets of the first order

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1 Introduction

One of the most interesting questions of the classical differential geometry which has appeared at studying of semi-Hamiltonian systems of hydrodynamical type is the description of the surfaces admitting not trivial deformations with preservation of principal directions and principal curvatures. Then the number of essential parameters on which such deformations depend, is actually equal to number various local Hamiltonian structures of corresponding system of hydrodynamical type [6]. Such local Hamiltonian structures are determined by differential-geometrical Poisson brackets of the first order (see [10]). In the same paper a bi-Hamiltonian structure of the system of averaged one-phase solutions of KdV have been considered. Later multi-Hamiltonian structures of systems of hydrodynamical type were studied in [12], [13]. The given work is devoted to the description of bi-Hamiltonian structures in language of orthogonal curvilinear coordinate nets. The alternative approach (by the method of an inverse scattering transform) has been used in [16]. In [5] it has been shown, that the description of pairs compatible differential-geometric Poisson brackets of the first order is equivalent to the solution of the over-determined system on rotation coefficients β_{ik} of orthogonal curvilinear coordi-

nate nets

$$\partial_i\beta_{jk}=\beta_{ji}\beta_{ik},\quad i\neq j\neq k,$$
$$\partial_i\beta_{ik}+\partial_k\beta_{ki}+\textstyle\sum_{m\neq i,k}\beta_{mi}\beta_{mk}=0,\qquad i\neq k,\tag{1}$$
$$\eta_i\partial_i\beta_{ik}+\eta_k\partial_k\beta_{ki}+\tfrac{1}{2}\eta_i'\beta_{ik}+\tfrac{1}{2}\eta_k'\beta_{ki}+\textstyle\sum_{m\neq i,k}\eta_m\beta_{mi}\beta_{mk}=0,\qquad i\neq k,$$

where $\eta_i(r^i)$ are some functions, $\partial_k\equiv\partial/\partial r^k$, $k=1,\,2,\,...,\,N$. The general case $\eta_i(r^i)\neq$ const and particular case $\eta_i(r^i)=c_i=$ const were described in [5], where it has been shown, that (1) is an integrable system. In that specific case $\eta_i(r^i)=c_i=$ const the system of equations

$$\partial_i\beta_{jk}=\beta_{ji}\beta_{ik},\quad i\neq j\neq k,$$
$$\partial_i\beta_{ik}+\partial_k\beta_{ki}+\textstyle\sum_{m\neq i,k}\beta_{mi}\beta_{mk}=0,\qquad i\neq k,\tag{2}$$
$$c_i\partial_i\beta_{ik}+c_k\partial_k\beta_{ki}+\textstyle\sum_{m\neq i,k}c_m\beta_{mi}\beta_{mk}=0,\qquad i\neq k,$$

when $N=3$ was reduced to three pairwise commuting non-evolution equations (see [5])

$$q_{xt}=\operatorname{ch}q\sqrt{1-q_x^2},\quad q_{yt}=\operatorname{sh}q\sqrt{1-q_y^2},\qquad q_{xy}=-\sqrt{(1-q_x^2)(1-q_y^2)},$$

which are nothing but two modified Sin-Gordon equations and degenerated twice modified Sin-Gordon equation, respectively. In this paper generalization of this system (i.e. $\eta_i(r^i)\neq$ const) is presented for bi-Hamiltonian structure where one metric is flat, and the second one has curvature of co-dimension 1.

2 Flat case

Compatibility condition of the linear system in partial derivatives

$$\partial_iH_k=\beta_{ik}H_i,\qquad i\neq k\tag{3}$$

is the same as compatibility condition for conjugated linear system

$$\partial_i\psi_k=\beta_{ki}\psi_i,\qquad i\neq k.\tag{4}$$

This yields the nonlinear multi-dimensional integrable N-wave system (see for instance, [4], [9])

$$\partial_i\beta_{jk}=\beta_{ji}\beta_{ik},\quad i\neq j\neq k.\tag{5}$$

Any pair of solutions of the first linear system $H_i^{(1)}$, $H_i^{(2)}$ generates hydrodynamic type system integrable by the generalized hodograph method [14]

$$r_t^i=\frac{H_i^{(2)}}{H_i^{(1)}}r_x^i,\qquad i=1,\,2,\,...,\,N,$$

written in Riemann invariants r^i. Particular solutions $\psi_i^{(k)}$ of the second linear system (4) determine densities a^k and fluxes c^k of conservation laws

$$a_t^k = \partial_x c^k(\mathbf{a}),$$

where

$$da^k = \sum_{m=1}^{N} \psi_m^{(k)} H_m^{(1)} dr^m, \qquad dc^k = \sum_{m=1}^{N} \psi_m^{(k)} H_m^{(2)} dr^m.$$

Existence of the linear differential operator of the first order connecting solutions of both linear systems (see for instance, [13])

$$H_i = \partial_i \psi_i + \sum_{m \neq i} \beta_{mi} \psi_m \tag{6}$$

is equivalent to existence of local Hamiltonian structure

$$a_t^i = \partial_x [g^{ik} \frac{\partial h}{\partial a^k}],$$

where flat coordinates a^k are determined by "null" solutions of the second linear system

$$0 = \partial_i \psi_i^{(k)} + \sum_{m \neq i} \beta_{mi} \psi_m^{(k)}, \qquad k = 1,\, 2,\, ...\, ,\, N,$$

and non-degenerate symmetric metric

$$ds^2 = \sum_{m=1}^{N} \left(H_m^{(1)} dr^m \right)^2 = g_{jk} da^j da^k,$$

which is constant in flat coordinates a^k

$$g^{jk} = \sum_{m=1}^{N} \psi_m^{(j)} \psi_m^{(k)} = \mathrm{const}\,.$$

Existence of such operator (6) imposes the restriction on rotation coefficients of orthogonal curvilinear coordinate nets β_{ik}

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i,k} \beta_{mi} \beta_{mk} = 0, \qquad i \neq k, \tag{7}$$

well-known in differential geometry as the Gauss equation, in above case it fixes the metric of zero curvature.

It is easy to see that N-wave system is invariant under the transformation $R^i = R^i(r^i)$, $\bar{H}_m^{(1)} = H_m^{(1)} \mu_i^{-1/2}(r^i)$, where $\mu_i(r^i)$ and $R^i(r^i)$ are arbitrary functions of a single variable. If Riemann invariants are fixed ($R^i(r^i) \equiv r^i$), then "flatness" condition (7) is no longer valid,

then solutions of the conjugate problem (4) transform in accordance with $\bar{\psi}_i = \psi_i \mu_i^{1/2}(r^i)$; if $\mu_i(r^i) = R^{i\prime 2}(r^i)$, then a metric preserves "flatness"

$$ds^2 = \sum_{m=1}^{N} \left(H_m^{(1)} dr^m \right)^2 = \sum_{m=1}^{N} \left(\bar{H}_m^{(1)} dR^m \right)^2 ,$$

and solutions of the conjugate problem (4) are the same

$$da^k = \sum_{m=1}^{N} \psi_m^{(k)} H_m^{(1)} dr^m = \sum_{m=1}^{N} \bar{\psi}_m^{(k)} \bar{H}_m^{(1)} dR^m .$$

Thus, infinite number of metrics

$$ds^2 = \sum_{m=1}^{N} \mu_m^{-1}(r^m) \left(H_m^{(1)} dr^m \right)^2 , \tag{8}$$

are connected to each system of hydrodynamic type and only for finite number of values $\mu_i(r^i)$ (no more than $N+1$, see [7]) the"flatness" condition is satisfied. In this paper the case is considered, when just two distinct values $\mu_i(r^i)$ determine flat metrics. Since coefficients of the diagonal metric H_i^2 are determined up to multiplication on an arbitrary function of corresponding Riemann invariant, then the first flat metric always can be chosen so that one of values $\mu_i(r^i)$ can be fixed to 1. Compatible pairs of local Hamiltonian structures determined by such metrics studied for instance in [16]. Such pairs are compatible if their arbitrary linear combination also is local Hamiltonian structure. Thus, the integrable nonlinear system (see [5])

$$\bar{\partial}_i \bar{\beta}_{jk} = \bar{\beta}_{ji} \bar{\beta}_{ik}, \quad i \neq j \neq k, \tag{9}$$

$$(\lambda + R^i)\bar{\partial}_i \bar{\beta}_{ik} + (\lambda + R^k)\bar{\partial}_k \bar{\beta}_{ki} + \frac{1}{2}(\bar{\beta}_{ik} + \bar{\beta}_{ki}) + \sum_{m \neq i,k} (\lambda + R^m) \bar{\beta}_{mi} \bar{\beta}_{mk} = 0, \quad i \neq k.$$

is a compatibility condition of the linear system

$$\begin{gathered} \bar{\partial}_i \bar{\psi}_j^{(k)} = \bar{\beta}_{ji} \bar{\psi}_i^{(k)}, \quad i \neq j, \\ 0 = (\lambda + R^i)\bar{\partial}_i \bar{\psi}_i^{(k)} + \tfrac{1}{2} \bar{\psi}_i^{(k)} + \textstyle\sum_{m \neq i} (\lambda + R^m) \bar{\beta}_{mi} \bar{\psi}_m^{(k)}, \quad k = 1, 2, \dots, N, \end{gathered} \tag{10}$$

where $\bar{\partial} = \partial / \partial R^i$, $\bar{\beta}_{ik}(\mathbf{R}) = \beta_{ik}(\mathbf{r}) / R^{i\prime}(r^i)$, $\bar{\psi}_j^{(k)} = \psi_j^{(k)}$, λ is an arbitrary constant of the flat metric $\tilde{g}^{ii}(\mathbf{r}) = (\lambda + R^i) g^{ii}$.

Thus, we have proved following

Lemma: The reduction of the N-wave system (5)

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \textstyle\sum_{m \neq i,k} \beta_{mi} \beta_{mk} = 0, \quad i \neq k,$$

$$\eta_i(r^i)\partial_i \beta_{ik} + \eta_k(r^k)\partial_k \beta_{ki} + \tfrac{1}{2}\eta_i'(r^i)\beta_{ik} + \tfrac{1}{2}\eta_k'(r^k)\beta_{ik} + \textstyle\sum_{m \neq i,k} \eta_m(r^m) \beta_{mi} \beta_{mk} = 0, \quad i \neq k,$$

where $\eta_i(r^i)$ is a second solution $\mu_i(r^i)$, is resulted in (9) by scaling $H_i(r) = \eta_i'(r^i)\tilde{H}_i(R)$, $\partial/\partial r^i = \eta_i'(r^i)\partial/\partial R^i$, $\eta_i(r^i) = \tilde{\eta}_i(R^i) \equiv R^i$, if $\eta_i(r^i) \neq c_i = \text{const}$.

Thus, in general case bi-Hamiltonian structures of hydrodynamic type systems are described by solutions of the integrable system

$$\partial_i\beta_{jk} = \beta_{ji}\beta_{ik}, \qquad i \neq j \neq k,$$

$$\partial_i\beta_{ik} + \partial_k\beta_{ki} + \textstyle\sum_{m\neq i,k}\beta_{mi}\beta_{mk} = 0, \qquad i \neq k,$$

$$r^i\partial_i\beta_{ik} + r^k\partial_k\beta_{ki} + \tfrac{1}{2}\beta_{ik} + \tfrac{1}{2}\beta_{ki} + \textstyle\sum_{m\neq i,k} r^m\beta_{mi}\beta_{mk} = 0, \qquad i \neq k.$$

Its spectral problem (10)

$$\partial_i\psi_j^{(k)} = \beta_{ji}\psi_i^{(k)}, \qquad i \neq j,$$

$$0 = (\lambda + r^i)\partial_i\psi_i^{(k)} + \tfrac{1}{2}\psi_i^{(k)} + \textstyle\sum_{m\neq i}(\lambda + r^m)\beta_{mi}\psi_m^{(k)}, \qquad k = 1,\, 2,\, ...,\, N$$

is nothing but N commuting systems of *ordinary* differential equations with respect to every independent variable (Riemann invariant r^i).

Remark: This multi-dimensional spectral problem has a multi-dimensional analog of Wronskian: $N(N+1)/2$ "first integrals" are constraints

$$g^{sn} = \sum_{m=1}^{N}(\lambda + r^m)\psi_m^{(s)}\psi_m^{(n)} = \text{const}, \tag{11}$$

which are nothing but metric coefficients in flat coordinates of *mixed* local Hamiltonian structure.

Following G. Darboux for further consideration we should determine "first integrals" of the Gauss equation (see (7)).

Definition: A *scalar* potential of rotation coefficients of conjugate curvilinear coordinate nets V is a such function, whose mixed second derivatives are

$$V_{ik} = \beta_{ik}\beta_{ki}, \qquad i \neq k.$$

Definition: A *vector* potential of rotation coefficients of conjugate curvilinear coordinate nets S_k is a such function, whose mixed first derivatives are

$$\partial_i S_k = \beta_{ik}\partial_k\beta_{ki}, \qquad i \neq k.$$

G. Darboux have proved (see [3]), that the Gauss equation, written via rotation coefficients of conjugate curvilinear coordinate nets (7), has N "first integrals"

$$S_i + \frac{1}{2}\sum_{m\neq i}\beta_{mi}^2 = n_i(r^i), \tag{12}$$

where $n_i(r^i)$ are arbitrary functions ("integration factors"). Similar zero curvature condition

$$\eta_i(r^i)\partial_i\beta_{ik} + \eta_k(r^k)\partial_k\beta_{ki} + \frac{1}{2}\eta_i'(r^i)\beta_{ik} + \frac{1}{2}\eta_k'(r^k)\beta_{ik} + \sum_{m\neq i,k}\eta_m(r^m)\beta_{mi}\beta_{mk} = 0, \qquad i \neq k$$

for metric $\eta_i(r^i)g^{ii}$ has N "first integrals"

$$\eta_i S_i + \frac{1}{2}\eta_i' V_i + \frac{1}{2}\sum_{m\neq i}\eta_m \beta_{mi}^2 = k_i(r^i), \tag{13}$$

where $k_i(r^i)$ are another arbitrary functions.

2.1 Particular case, $\eta_i(r^i) = c_i = \text{const}$

This particular case was considered in [5]. The spectral problem

$$\partial_i \psi_j^{(k)} = \beta_{ji}\psi_i^{(k)}, \qquad i \neq j,$$

$$0 = (\lambda + c_i)\partial_i \psi_i^{(k)} + \textstyle\sum_{m\neq i}(\lambda + c_m)\beta_{mi}\psi_m^{(k)}, \qquad k = 1,\, 2,\, \ldots\, ,\, N,$$

determines the nonlinear system (2)

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k,$$

$$\partial_i \beta_{ik} = -\textstyle\sum_{m\neq i,k} \frac{c_m - c_k}{c_i - c_k}\beta_{mi}\beta_{mk}, \qquad i \neq k.$$

It easy to see that the consequences of (12) and (13)

$$S_i + \tfrac{1}{2}\textstyle\sum_{m\neq i}\beta_{mi}^2 = n_i(r^i),$$

$$c_i S_i + \tfrac{1}{2}\textstyle\sum_{m\neq i} c_m \beta_{mi}^2 = k_i(r^i)$$

for particular case $\eta_i(r^i) = c_i = \text{const}$ are the constraints

$$\sum_{m\neq i}(c_m - c_i)\beta_{mi}^2 = 2[k_i(r^i) - c_i n_i(r^i)].$$

Thus, if $N = 3$, then above system

$$\begin{aligned}
(c_2 - c_1)\beta_{21}^2 + (c_3 - c_1)\beta_{31}^2 &= l_1(r^1), \\
(c_1 - c_2)\beta_{12}^2 + (c_3 - c_2)\beta_{32}^2 &= l_2(r^2), \\
(c_1 - c_3)\beta_{13}^2 + (c_2 - c_3)\beta_{23}^2 &= l_3(r^3),
\end{aligned}$$

can be parameterized

$$\begin{aligned}
\beta_{21} &= \sqrt{\frac{l_1(r^1)}{c_2 - c_1}}\cosh u, \qquad \beta_{31} = \sqrt{\frac{l_1(r^1)}{c_1 - c_3}}\sinh u, \\
\beta_{12} &= \sqrt{\frac{l_2(r^2)}{c_1 - c_2}}\cosh v, \qquad \beta_{32} = \sqrt{\frac{l_2(r^2)}{c_2 - c_3}}\sinh v, \\
\beta_{13} &= \sqrt{\frac{l_3(r^3)}{c_1 - c_3}}\cosh w, \qquad \beta_{23} = \sqrt{\frac{l_3(r^3)}{c_3 - c_2}}\sinh w.
\end{aligned}$$

Then the 3-wave system has the form

$$\begin{aligned}
\frac{1}{\sqrt{l_1(r^1)}}\partial_1 w &= \sqrt{\frac{c_2-c_3}{(c_1-c_2)(c_1-c_3)}}\cosh u, \quad \frac{1}{\sqrt{l_3(r^3)}}\partial_3 u = \sqrt{\frac{c_1-c_2}{(c_1-c_3)(c_2-c_3)}}\sinh w,\\
\frac{1}{\sqrt{l_1(r^1)}}\partial_1 \upsilon &= \sqrt{\frac{c_2-c_3}{(c_1-c_2)(c_1-c_3)}}\sinh u, \quad \frac{1}{\sqrt{l_2(r^2)}}\partial_2 u = \sqrt{\frac{c_1-c_3}{(c_2-c_1)(c_2-c_3)}}\sinh \upsilon,\\
\frac{1}{\sqrt{l_2(r^2)}}\partial_2 w &= \sqrt{\frac{c_1-c_3}{(c_1-c_2)(c_3-c_2)}}\cosh \upsilon, \quad \frac{1}{\sqrt{l_3(r^3)}}\partial_3 \upsilon = \sqrt{\frac{c_1-c_2}{(c_1-c_3)(c_2-c_3)}}\cosh w.
\end{aligned}$$

Introducing new independent variables (scaling Riemann invariants)

$$\begin{aligned}
p &= \sqrt{\frac{c_2-c_3}{(c_1-c_2)(c_1-c_3)}}\int\sqrt{l_1(r^1)}dr^1,\\
q &= i\sqrt{\frac{c_1-c_3}{(c_1-c_2)(c_2-c_3)}}\int\sqrt{l_2(r^2)}dr^2,\\
s &= \sqrt{\frac{c_1-c_2}{(c_1-c_3)(c_2-c_3)}}\int\sqrt{l_3(r^3)}dr^3,
\end{aligned}$$

finally one can obtain the system (see [5])

$$\begin{aligned}
\partial_p w &= \cosh u, \quad \partial_s u = \sinh w,\\
\partial_p \upsilon &= \sinh u, \quad \partial_q u = \sinh \upsilon,\\
\partial_q w &= \cosh \upsilon, \quad \partial_s \upsilon = \cosh w.
\end{aligned}$$

For instance, eliminating field variables υ and w ($\upsilon = \operatorname{arcsinh} u_q,\ \ w = \operatorname{arcsinh} u_s$), one can obtain two modified sinh-Gordon equations

$$u_{pq} = \sinh u\sqrt{u_q^2+1}, \qquad u_{sp} = \cosh u\sqrt{u_s^2+1}$$

and the degenerated twice modified sinh-Gordon equation ([1])

$$u_{sq} = \sqrt{(u_q^2+1)(u_s^2+1)}.$$

Such equations can be obtained as well as with respect to other field υ and w

$$\begin{aligned}
\upsilon_{pq} &= \sinh\upsilon\sqrt{\upsilon_p^2+1}, \quad \upsilon_{sq} = \cosh\upsilon\sqrt{\upsilon_s^2-1}, \quad \upsilon_{sp} = \sqrt{(\upsilon_p^2+1)(\upsilon_s^2-1)},\\
w_{sq} &= \cosh w\sqrt{w_q^2-1}, \quad w_{sp} = \sinh w\sqrt{w_p^2-1}, \quad w_{pq} = \sqrt{(w_p^2-1)(w_q^2-1)}.
\end{aligned}$$

The substitution

$$z = \upsilon + w = w + \ln[w_q + \sqrt{w_q^2-1}]$$

transforms these equations into

$$z_{sq} = \sinh z, \qquad z_{ssp} = z_s\sqrt{z_p^2+z_{sp}^2} - z_p. \tag{14}$$

The first one is well-known Bonnet equation (the sinh-Gordon equation), the second one is the next commuting flow from a hierarchy of the potential modified KdV equation (see [15])

$$z_\tau = z_{sss} - \frac{1}{2} z_s^3.$$

Thus, a spectral problem for a *triple* of the modified sinh-Gordon equations has a form

$$\begin{pmatrix} \psi \\ \bar\psi \\ \tilde\psi \end{pmatrix}_p = \begin{pmatrix} 0 & (1-\frac{1}{\zeta})\cosh u & \frac{1}{\zeta}\sinh u \\ \cosh u & 0 & 0 \\ \sinh u & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \bar\psi \\ \tilde\psi \end{pmatrix},$$

$$\begin{pmatrix} \psi \\ \bar\psi \\ \tilde\psi \end{pmatrix}_q = \begin{pmatrix} 0 & \cosh v & 0 \\ -\frac{\zeta}{1-\zeta}\cosh v & 0 & \frac{1}{1-\zeta}\sinh v \\ 0 & \sinh v & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \bar\psi \\ \tilde\psi \end{pmatrix},$$

$$\begin{pmatrix} \psi \\ \bar\psi \\ \tilde\psi \end{pmatrix}_s = \begin{pmatrix} 0 & 0 & \cosh w \\ 0 & 0 & \sinh w \\ \zeta\cosh w & (1-\zeta)\sinh w & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \bar\psi \\ \tilde\psi \end{pmatrix},$$

where $\psi_1 = \sqrt{c_2 - c_3}\psi$, $\psi_2 = \sqrt{c_3 - c_1}\bar\psi$, $\psi_3 = \sqrt{c_1 - c_2}\tilde\psi$, a spectral parameter is

$$\zeta = -\frac{(\lambda + c_1)(c_2 - c_3)}{(\lambda + c_3)(c_1 - c_2)}.$$

Remark: Each above linear system can be reduced to well-known spectral problem 2x2 for the modified sinh-Gordon equation (see [1]) by transformations described in [17]. This becomes obvious if take into account that all three above linear systems 3x3 have constraint (see (11))

$$\zeta\psi^2 + (1-\zeta)\bar\psi^2 - \tilde\psi^2 = \text{const}\,.$$

It means that the functions ψ, $\bar\psi$, $\tilde\psi$ are "squares of eigenfunctions" for corresponding spectral transform 2x2 (see [17]). Also, every above linear system 3x3 can be written as a *scalar* spectral problem of the third order. For instance, the third linear system is equivalent to the third order linear equation

$$\tilde\psi_{sss} - \frac{w_{ss}}{w_s}\tilde\psi_{ss} - (\zeta + \sinh^2 w + w_s^2)\tilde\psi_s + [(\zeta + \sinh^2 w)\frac{w_{ss}}{w_s} - 3w_s \sinh w \cosh w]\tilde\psi = 0. \quad (15)$$

At the same time, the well-known fact is (see for instance, [8]) that the Yajima-Oikawa hierarchy (the Yajima-Oikawa system also is known as the *long-short resonance*) is associated with the spectral problem

$$\hat L\tilde\psi = \zeta\tilde\psi, \quad (16)$$

where

$$\hat L = \partial_s^2 - a_0 + a_1\partial_s^{-1}a_2. \quad (17)$$

Substituting (17) in (16) and comparing with (15), one can obtain expressions for the coefficients a_k

$$a_0 = \sinh^2 w + w_s^2, \qquad a_1 = w_s, \qquad a_2 = w_{ss} - \sinh w\cosh w.$$

That means, that the sinh-Gordon equation is embedded not only to the KdV hierarchy but also to the Yajima-Oikawa hierarchy. In other words: the pseudo-differential Manin-Sato operator

$$\hat{L} = \partial_s + A_0\partial_s^{-1} + A_1\partial_s^{-2} + ...$$

associated with the KP hierarchy has finite-component reductions [9]

$$\hat{L}^n = \partial_s^n + A_{0,n}\partial_s^{n-2} + ... + A_{n-2,n} + \sum_{k=1}^{N} B_{k,n}\partial_s^{-1}C_{k,n},$$

where N is an arbitrary natural number. Thus the KP reduction determined by operator

$$\hat{L} = \partial_s^2 - (\sinh^2 w + w_s^2) + w_s\partial_s^{-1}(w_{ss} - \sinh w \cosh w),$$

describes three-component bi-Hamiltonian structures of hydrodynamic type systems.

Remark: One can introduce field variable $c = z_p$, then re-scale independent variables $\partial_s \to \varepsilon\partial_s,\ \ \partial_p \to \varepsilon^{-1}\partial_p$, then a second equation from (14) has a form

$$\varepsilon^2\partial_s c = \partial_p \frac{c + \varepsilon^2 c_{ss}}{\sqrt{c^2 + \varepsilon^2 c_s^2}}.$$

A limit of this equation with respect to the parameter ε when $\varepsilon \to 0$ yields

$$\partial_s c = \partial_p\Big[\frac{c_{ss}}{c} - \frac{c_s^2}{2c^2}\Big],$$

which is nothing but again the sinh-Gordon equation

$$b_{sp} = \sinh b,$$

where $\ln c = b$.

Remark: The substitutions

$$z^1 = w \pm \upsilon, \qquad z^2 = \upsilon \pm u, \qquad z^3 = u \pm w$$

connect solutions of the sinh-Gordon equation

$$z^1_{sq} = \sinh z^1, \qquad z^2_{pq} = \sinh z^2, \qquad z^3_{sp} = \cosh z^3.$$

2.2 General case, $\eta_i(r^i) = r^i$

In general case $\eta_i(r^i) = r^i$ the spectral problem

$$\partial_i\psi_j^{(k)} = \beta_{ji}\psi_i^{(k)}, \qquad i \neq j,$$

$$0 = (\lambda + r^i)\partial_i\psi_i^{(k)} + \tfrac{1}{2}\psi_i^{(k)} + \textstyle\sum_{m\neq i}(\lambda + r^m)\beta_{mi}\psi_m^{(k)}, \qquad k = 1,\ 2,\ ...\ ,\ N,$$

determines the nonlinear system

$$\partial_i\beta_{jk}=\beta_{ji}\beta_{ik},\quad i\neq j\neq k,$$

$$\partial_i\beta_{ik}=\tfrac{1}{r^i-r^k}[-\tfrac{1}{2}(\beta_{ik}+\beta_{ki})+\textstyle\sum_{m\neq i,k}(r^k-r^m)\beta_{mi}\beta_{mk}],\qquad i\neq k.$$

It is easy to see that consequences of (12) and (13) yield another "first integrals"

$$\sum_{m\neq i}(r^i-r^m)\beta_{mi}^2=V_i.$$

For instance, if $N=3$, then above constraints

$$\begin{aligned}(r^1-r^2)\beta_{21}^2+(r^1-r^3)\beta_{31}^2&=V_1,\\(r^2-r^1)\beta_{12}^2+(r^2-r^3)\beta_{32}^2&=V_2,\\(r^3-r^1)\beta_{13}^2+(r^3-r^2)\beta_{23}^2&=V_3.\end{aligned}$$

describe a bi-Hamiltonian structure determined by the flat metrics g^{ii} and r^ig^{ii}.

Using parametrization

$$\begin{aligned}\beta_{21}&=\sqrt{\frac{V_1}{r^1-r^2}}\cosh u,\qquad \beta_{31}=\sqrt{\frac{V_1}{r^3-r^1}}\sinh u,\\ \beta_{32}&=\sqrt{\frac{V_2}{r^2-r^3}}\cosh \upsilon,\qquad \beta_{12}=\sqrt{\frac{V_2}{r^1-r^2}}\sinh \upsilon,\\ \beta_{13}&=\sqrt{\frac{V_3}{r^3-r^1}}\cosh w,\qquad \beta_{23}=\sqrt{\frac{V_3}{r^2-r^3}}\sinh w,\end{aligned}$$

the system (5) transforms into

$$\begin{aligned}\partial_1 w&=-\frac{\sqrt{V_1/V_3}}{2(r^3-r^1)}\sinh u\sinh w+\sqrt{\frac{r^2-r^3}{(r^1-r^2)(r^3-r^1)}}\sqrt{V_1}\cosh u,\\ \partial_3 u&=-\frac{\sqrt{V_3/V_1}}{2(r^3-r^1)}\cosh w\cosh u+\sqrt{\frac{r^1-r^2}{(r^3-r^1)(r^2-r^3)}}\sqrt{V_3}\sinh w,\\ V_{13}&=\frac{\sqrt{V_1V_3}}{r^3-r^1}\cosh w\sinh u,\end{aligned}$$

$$\begin{aligned}\partial_1 \upsilon&=-\frac{\sqrt{V_1/V_2}}{2(r^1-r^2)}\cosh u\cosh \upsilon+\sqrt{\frac{r^2-r^3}{(r^1-r^2)(r^3-r^1)}}\sqrt{V_1}\sinh u,\\ \partial_2 u&=-\frac{\sqrt{V_2/V_1}}{2(r^1-r^2)}\sinh \upsilon\sinh u+\sqrt{\frac{r^3-r^1}{(r^1-r^2)(r^2-r^3)}}\sqrt{V_2}\cosh \upsilon,\\ V_{12}&=\frac{\sqrt{V_1V_2}}{r^1-r^2}\sinh \upsilon\cosh u,\end{aligned}$$

$$
\begin{aligned}
\partial_2 w &= -\frac{\sqrt{V_2/V_3}}{2(r^2-r^3)}\cosh\upsilon\cosh w + \sqrt{\frac{r^3-r^1}{(r^1-r^2)(r^2-r^3)}}\sqrt{V_2}\sinh\upsilon,\\
\partial_3 \upsilon &= -\frac{\sqrt{V_3/V_2}}{2(r^2-r^3)}\sinh w\sinh\upsilon + \sqrt{\frac{r^1-r^2}{(r^3-r^1)(r^2-r^3)}}\sqrt{V_3}\cosh w,\\
V_{23} &= \frac{\sqrt{V_2V_3}}{r^2-r^3}\cosh\upsilon\sinh w,
\end{aligned}
$$

where $V_{ik}=\beta_{ik}\beta_{ki}$. Each above triple of equation depends from third independent variable as a parameter, which can be eliminated by shift along both other independent variables. Thus, every above system can be written in form

$$
\begin{aligned}
\partial_x w &= \frac{\sqrt{V_x/V_y}}{2(x-y)}\sinh u\sinh w + \sqrt{\frac{y}{x(x-y)}}\sqrt{V_x}\cosh u,\\
\partial_y u &= \frac{\sqrt{V_y/V_x}}{2(x-y)}\cosh w\cosh u + \sqrt{\frac{x}{y(x-y)}}\sqrt{V_y}\sinh w,\\
V_{xy} &= -\frac{\sqrt{V_xV_y}}{x-y}\cosh w\sinh u.
\end{aligned}
$$

3 Co-dimension 1

In previous section a flat case is considered, where relationship between two conjugate linear problems (3), (4) is determined by the linear differential operator of the first order (6). In general case of arbitrary curvature matric (8) such relationship becomes nonlocal (see for details [11])

$$
H_i=\partial_i\psi_i+\sum_{m\neq i}\beta_{mi}\psi_m+\sum_{k=1}^{M}\sum_{n=1}^{M}\varepsilon_{kn}H_i^{(k)}a^{(n)}, \tag{18}
$$

where $\partial_i a^{(n)}=\psi_i H_i^{(n)}$, ε_{kn} are constant symmetric non-degenerate matrix, and a number of particular solutions $H_i^{(n)}$ of (3) is determined by the Gauss equation (see [11])

$$
\partial_i\beta_{ij}+\partial_j\beta_{ji}+\sum_{m\neq i,j}\beta_{mi}\beta_{mj}=\sum_{k=1}^{M}\sum_{n=1}^{M}\varepsilon_{kn}H_i^{(k)}H_j^{(n)},\qquad i\neq j.
$$

Thus, in general case the description of pairs of nonlocal differential operators (18) (similar to the flat case) is equivalent to the system

$$
\begin{gathered}
\partial_i\beta_{jk}=\beta_{ji}\beta_{ik},\qquad i\neq j\neq k,\\
\partial_i\beta_{ij}+\partial_j\beta_{ji}+\textstyle\sum_{m\neq i,j}\beta_{mi}\beta_{mj}=\sum_{k=1}^{M}\sum_{n=1}^{M}\varepsilon_{kn}H_i^{(k)}H_j^{(n)},\qquad i\neq j,\\
r^i\partial_i\beta_{ij}+r^j\partial_j\beta_{ji}+\tfrac12\beta_{ij}+\tfrac12\beta_{ji}+\textstyle\sum_{m\neq i,j}r^m\beta_{mi}\beta_{mj}=\sum_{k=1}^{\tilde M}\sum_{n=1}^{\tilde M}\tilde\varepsilon_{kn}\tilde H_i^{(k)}\tilde H_j^{(n)},\qquad i\neq j.
\end{gathered}
$$

In case of non-flat metric $\eta_i(r^i)g^{ii}$ the Gauss equation

$$\eta_i(r^i)\partial_i\beta_{ij}+\eta_j(r^j)\partial_j\beta_{ji}+\frac{1}{2}\eta_i'(r^i)\beta_{ij}+\frac{1}{2}\eta_j'(r^j)\beta_{ij}+\sum_{m\neq i,j}\eta_m(r^m)\beta_{mi}\beta_{mj}=\sum_{k=1}^{M}\sum_{n=1}^{M}\varepsilon_{kn}H_i^{(k)}H_j^{(n)}, \qquad i\neq k$$

has N "first integrals"

$$\eta_i S_i+\frac{1}{2}\eta_i' V_i+\frac{1}{2}\sum_{m\neq i}\eta_m\beta_{mi}^2=\frac{1}{2}\sum_{k=1}^{M}\sum_{n=1}^{M}\varepsilon_{kn}H_i^{(k)}H_i^{(n)}+k_i(r^i),$$

where $k_i(r^i)$ are some functions.

In this section we restrict our consideration on two simplest cases: a flat metric g^{ii} and a metric $\tilde{g}^{ii}$ of co-dimension 1, when $\tilde{g}^{ii}=c_i g^{ii}$ or $\tilde{g}^{ii}=r^i g^{ii}$. In the first case the system

$$\partial_i\beta_{jk}=\beta_{ji}\beta_{ik}, \quad i\neq j\neq k,$$

$$\partial_i\beta_{ij}+\partial_j\beta_{ji}+\textstyle\sum_{m\neq i,j}\beta_{mi}\beta_{mj}=0, \qquad i\neq j, \tag{19}$$

$$c_i\partial_i\beta_{ij}+c_j\partial_j\beta_{ji}+\textstyle\sum_{m\neq i,j}c_m\beta_{mi}\beta_{mj}=H_iH_j, \qquad i\neq j,$$

is a result of a compatibility condition of the linear system (see [16])

$$\partial_i\psi_j=\beta_{ji}\psi_i, \qquad i\neq j,$$

$$0=(\lambda+c_i)\partial_i\psi_i+\textstyle\sum_{m\neq i}(\lambda+c_m)\beta_{mi}\psi_m+H_ia,$$

$$\partial_i a=H_i\psi_i.$$

In the second case the system

$$\partial_i\beta_{jk}=\beta_{ji}\beta_{ik}, \quad i\neq j\neq k,$$

$$\partial_i\beta_{ij}+\partial_j\beta_{ji}+\textstyle\sum_{m\neq i,j}\beta_{mi}\beta_{mj}=0, \qquad i\neq j, \tag{20}$$

$$r^i\partial_i\beta_{ij}+r^j\partial_j\beta_{ji}+\tfrac{1}{2}\beta_{ij}+\tfrac{1}{2}\beta_{ji}+\textstyle\sum_{m\neq i,j}r^m\beta_{mi}\beta_{mj}=H_iH_j, \qquad i\neq j,$$

is a result of a compatibility condition of the linear system (see [16])

$$\partial_i\psi_j=\beta_{ji}\psi_i, \qquad i\neq j,$$

$$0=(\lambda+r^i)\partial_i\psi_i+\tfrac{1}{2}\psi_i+\textstyle\sum_{m\neq i}(\lambda+r^m)\beta_{mi}\psi_m+H_ia,$$

$$\partial_i a=H_i\psi_i.$$

3.1 Particular case, $\eta_i(r^i) = c_i = \mathrm{const}$

As well as in the flat case the nonlinear system (19)

$$\partial_i\beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k,$$

$$\partial_i\beta_{ik} = \tfrac{1}{c_i - c_k}[H_iH_k + \textstyle\sum_{m\neq i,k}(c_k - c_m)\beta_{mi}\beta_{mk}], \qquad i \neq k$$

has the constraints

$$S_i + \tfrac{1}{2}\textstyle\sum_{m\neq i}\beta_{mi}^2 = n_i(r^i),$$

$$c_iS_i + \tfrac{1}{2}\textstyle\sum_{m\neq i}c_m\beta_{mi}^2 = \tfrac{1}{2}H_i^2 + k_i(r^i).$$

Thus, one has

$$\sum_{m\neq i}(c_i - c_m)\beta_{mi}^2 + H_i^2 = 2[c_in_i(r^i) - k_i(r^i)].$$

When $N = 3$, then above system has a form

$$\begin{array}{rcl}
(c_1 - c_2)\beta_{21}^2 + (c_1 - c_3)\beta_{31}^2 + H_1^2 &=& l_1(r^1), \\
(c_2 - c_1)\beta_{12}^2 + (c_2 - c_3)\beta_{32}^2 + H_2^2 &=& l_2(r^2), \\
(c_3 - c_1)\beta_{13}^2 + (c_3 - c_2)\beta_{23}^2 + H_3^2 &=& l_3(r^3).
\end{array}$$

Introducing new independent variables (re-scaling Riemann invariants)

$$\begin{array}{rcl}
p &=& i\sqrt{\dfrac{c_2 - c_3}{(c_1 - c_2)(c_1 - c_3)}}\displaystyle\int\sqrt{l_1(r^1)}dr^1, \\
q &=& \sqrt{\dfrac{c_1 - c_3}{(c_1 - c_2)(c_2 - c_3)}}\displaystyle\int\sqrt{l_2(r^2)}dr^2, \\
s &=& \sqrt{\dfrac{c_1 - c_2}{(c_1 - c_3)(c_2 - c_3)}}\displaystyle\int\sqrt{l_3(r^3)}dr^3,
\end{array}$$

and new dependent functions

$$\begin{array}{rcl}
H_1 &=& \sqrt{l_1}R_1, \quad H_2 = \sqrt{l_2}S_2, \quad H_3 = \sqrt{l_3}P_3, \\
\beta_{13} &=& i\sqrt{\dfrac{l_3}{c_1 - c_3}}P_2, \quad \beta_{31} = \sqrt{\dfrac{l_1}{c_1 - c_3}}R_2, \quad \beta_{21} = \sqrt{\dfrac{l_1}{c_1 - c_2}}R_3, \\
\beta_{23} &=& -i\sqrt{\dfrac{l_3}{c_2 - c_3}}P_1, \quad \beta_{12} = i\sqrt{\dfrac{l_2}{c_1 - c_2}}S_3, \quad \beta_{32} = \sqrt{\dfrac{l_2}{c_2 - c_3}}S_1,
\end{array}$$

the system (5) can be written in form pairwise commuting hyperbolic systems

$$\begin{aligned}
\partial_p P_1 &= iR_3P_2, & \partial_s R_1 &= \frac{\Delta}{\sqrt{1-\Delta^2}}P_3R_2,\\
\partial_p P_2 &= -iP_1R_3 - \frac{\sqrt{1-\Delta^2}}{\Delta}P_3R_1, & \partial_s R_2 &= iR_3P_1 - \frac{\Delta}{\sqrt{1-\Delta^2}}R_1P_3,\\
\partial_p P_3 &= \frac{\sqrt{1-\Delta^2}}{\Delta}R_1P_2, & \partial_s R_3 &= -iP_1R_2,
\end{aligned}$$

$$\begin{aligned}
\partial_p S_1 &= R_2S_3, & \partial_q R_1 &= \Delta S_2R_3,\\
\partial_p S_2 &= \frac{1}{\Delta}R_1S_3, & \partial_q R_2 &= S_1R_3,\\
\partial_p S_3 &= -R_2S_1 - \frac{1}{\Delta}R_1S_2, & \partial_q R_3 &= -S_1R_2 - \Delta S_2R_1,
\end{aligned}$$

$$\begin{aligned}
\partial_q P_1 &= iS_3P_2 + i\sqrt{1-\Delta^2}P_3S_2, & \partial_s S_1 &= -S_3P_2 - \frac{1}{\sqrt{1-\Delta^2}}S_2P_3,\\
\partial_q P_2 &= -iS_3P_1, & \partial_s S_2 &= \frac{1}{\sqrt{1-\Delta^2}}P_3S_1,\\
\partial_q P_3 &= -i\sqrt{1-\Delta^2}S_2P_1, & \partial_s S_3 &= P_2S_1.
\end{aligned}$$

where

$$\Delta^2 = \frac{c_2-c_3}{c_1-c_3}, \qquad P_1^2+P_2^2+P_3^2=1, \qquad R_1^2+R_2^2+R_3^2=1, \qquad S_1^2+S_2^2+S_3^2=1.$$

Each above system is nothing but a some reduction of the famous Cherednik model of chiral fields related with hierarchy of the Heisenberg magnet (see [2]).

3.2 General case, $\eta_i(r^i) = r^i$

In general case the nonlinear system (20)

$$\begin{aligned}
\partial_i\beta_{jk} &= \beta_{ji}\beta_{ik}, \quad i\neq j\neq k,\\
\partial_i\beta_{ik} &= \tfrac{1}{r^i-r^k}[H_iH_k - \tfrac12(\beta_{ik}+\beta_{ki}) + \textstyle\sum_{m\neq i,k}(r^k-r^m)\beta_{mi}\beta_{mk}], \qquad i\neq k.
\end{aligned} \tag{21}$$

has a set of “first integrals”

$$S_i + \tfrac12\textstyle\sum_{m\neq i}\beta_{mi}^2 = n_i(r^i),$$

$$r^iS_i + \tfrac12 V_i + \tfrac12\textstyle\sum_{m\neq i} r^m\beta_{mi}^2 = \tfrac12 H_i^2 + k_i(r^i).$$

Then we have

$$V_i + \sum_{m\neq i}(r^m-r^i)\beta_{mi}^2 = H_i^2 + 2[k_i(r^i) - c_in_i(r^i)].$$

Thus, if $N = 3$, then the system of 18 equations (21), written in form of three pairwise commuting hyperbolic flows

$$\begin{array}{cc}
\partial_1\beta_{23} = \beta_{21}\beta_{13}, & \partial_3\beta_{21} = \beta_{23}\beta_{31}, \\
\partial_1 H_3 = \beta_{13}H_1, & \partial_3 H_1 = \beta_{31}H_3, \\
\partial_1\beta_{13} = \frac{H_1H_3}{r^1-r^3} - \frac{\beta_{13}+\beta_{31}}{2(r^1-r^3)} - \frac{r^2-r^3}{r^1-r^3}\beta_{21}\beta_{23}, & \partial_3\beta_{31} = -\frac{H_1H_3}{r^1-r^3} + \frac{\beta_{13}+\beta_{31}}{2(r^1-r^3)} - \frac{r^1-r^2}{r^1-r^3}\beta_{21}\beta_{23}, \\
\partial_1\beta_{32} = \beta_{31}\beta_{12}, & \partial_2\beta_{31} = \beta_{32}\beta_{21}, \\
\partial_1 H_2 = \beta_{12}H_1, & \partial_2 H_1 = \beta_{21}H_2, \\
\partial_1\beta_{12} = \frac{H_1H_2}{r^1-r^2} - \frac{\beta_{12}+\beta_{21}}{2(r^1-r^2)} + \frac{r^2-r^3}{r^1-r^2}\beta_{31}\beta_{32}, & \partial_2\beta_{21} = -\frac{H_1H_2}{r^1-r^2} + \frac{\beta_{12}+\beta_{21}}{2(r^1-r^2)} - \frac{r^1-r^3}{r^1-r^2}\beta_{31}\beta_{32}, \\
\partial_2\beta_{13} = \beta_{12}\beta_{23}, & \partial_3\beta_{12} = \beta_{13}\beta_{32}, \\
\partial_2 H_3 = \beta_{23}H_2, & \partial_3 H_2 = \beta_{32}H_3, \\
\partial_2\beta_{23} = \frac{H_2H_3}{r^2-r^3} - \frac{\beta_{23}+\beta_{32}}{2(r^2-r^3)} - \frac{r^1-r^3}{r^2-r^3}\beta_{12}\beta_{13}, & \partial_3\beta_{32} = -\frac{H_2H_3}{r^2-r^3} + \frac{\beta_{23}+\beta_{32}}{2(r^2-r^3)} + \frac{r^1-r^2}{r^2-r^3}\beta_{12}\beta_{13},
\end{array}$$

has three "first integrals"

$$\begin{aligned}
(r^1-r^2)\beta_{21}^2 + (r^1-r^3)\beta_{31}^2 + H_1^2 &= V_1, \\
(r^2-r^1)\beta_{12}^2 + (r^2-r^3)\beta_{32}^2 + H_2^2 &= V_2, \\
(r^3-r^1)\beta_{13}^2 + (r^3-r^2)\beta_{23}^2 + H_3^2 &= V_3.
\end{aligned}$$

Since this example and for higher dimensions and co-dimensions presence of "the first integrals" any more does not lead to an essential reduction of number of the equations in each of such hyperbolic systems. For example, at $N = 3$ from 18 equations in case of two flat metrics where ratios of corresponding diagonal coefficients are various constants, - there are 6 equations of the first order; if ratios of corresponding diagonal coefficients are some functions of Riemann invariants, – there are 9 equations of the first order; if one of metrics is flat, and the second one is of co-dimension 1, and ratios of corresponding diagonal coefficients are various constants, - there are 12 equations of the first order; and at last, if ratios of corresponding diagonal coefficients are some functions, - there are 15 equations of the first order (see explanation below), - in all other more complicated cases (any dimension of space N and any co-dimension M) the number of equations in system in involution will be more the of use of "first integrals", than in initial system of the equations. For example, in last above-stated example - the system has 18 equations. Use of 3 constraints on a first step decreases number of the equations up to 12. However, as constraints contain the first derivative of scalar potential V of rotation coefficients of conjugate curvilinear coordinate nets, it is necessary to add 3 more equations expressing second derivative from V through rotation coefficients of conjugate curvilinear coordinate nets. Thus, actually already in this example it is visible, that above system contains the same number of equations (as any equation of the second order is equivalent to a pair of the first order equations).

4 Conclusion

In this paper it has been shown, that the problem of classification of semi-Hamiltonian systems of hydrodynamical type with such additional and important properties as a flatness

of the diagonal metrics or metric's curvature of co-dimension 1 is reduced to the solution of such well-known integrable systems (by inverse scattering transform) as a negative flow of the Landau-Lifshitz equation (i.e. the Cherednik model) and generalization of the Ernst equation well known in Gravity and General Relativity (here a spectral problem depends explicitly of independent variable).

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References

[1] *A.B. Borisov, S.A. Zykov*, The dressing chain of discrete symmetries and the proliferation of nonlinear equations. Theor. Math. Phys. **115**, No. 2 (1998) 530–541.

[2] *I.V. Cherednik*, On the integrability of the two-dimensional asymmetric chiral O(3)-field and its quantum analogue. Yadernaya Fizika (in Russian) Soviet Journal of Nuclear Physics **33** (1981) 278-282.

[3] *G. Darboux,* Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. Paris, Gautier-Villar, 1910.

[4] *B.A. Dubrovin*, Geometry of 2D topological field theories, Lecture Notes in Math. **1620** Springer-Verlag (1996) 120-348.

[5] *E.V. Ferapontov*, Compatible Poisson brackets of hydrodynamic type, J. Phys. A: Math. Gen. **34** (2001) 1-12.

[6] *E.V. Ferapontov*, Hamiltonian systems of hydrodynamic type and their realization on hypersurfaces of a pseudoeuclidean space, Geom. Sbornik, VINITI **22** (1990) 59-96; (English transl. in Soviet J. of Math., **55** (1991) 1970-1995).

[7] *E.V. Ferapontov, M.V. Pavlov*, Quasiclassical limit of Coupled KdV equations. Riemann invariants and multi-Hamiltonian structure. Physica D **52** (1991), 211–219.

[8] *W. Oevel, W. Strampp*, Constrained KP hierarchy and Bi-Hamiltonian structures. Comm. Math. Phys. **157** (1993), 51-81.

[9] *A.Yu. Orlov*, Volterra operator algebra for zero curvature representation. Universality of KP. Proceedings of the III Potsdam-Kiev Workshop at Clarkson University, Potsdam, NY, USA, August, 1-11, 1991.

[10] *B.A. Dubrovin, S.P. Novikov*, Hydrodynamics of soliton lattices. *Soviet Scientific Reviews*, Section C: *Mathematical Physics Reviews* **9** Part 4. Harwood Academic Publishers GmbH, Yverdon (1993) 136 pp.

[11] *E.V. Ferapontov*, Nonlocal Hamiltonian operators of hydrodynamic type: Differential geometry and Applications, Amer. Math. Soc. Transl. (2) **170** (1995) 33-58.

[12] *M.V. Pavlov,* Elliptic coordinates and multi-Hamiltonian structures of hydrodynamic type systems. Russian Acad. Sci. Dokl. Math. **50** No. 3 (1995) 374-377. *M.V. Pavlov,* Multi-Hamiltonian structures of the Whitham equations. Russian Acad. Sci. Dokl. Math. **50** No. 2 (1995) 220-223.

[13] *M.V. Pavlov, S.P. Tsarev,* Three Hamiltonian structures of the Egorov hydrodynamic type systems. Funct. Anal. Appl. **37** No. 1 (2003) 32-45.

[14] *S.P. Tsarev,* On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, Soviet Math. Dokl. **31** (1985) 488–491. *S.P. Tsarev,* The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph transform, Math. USSR Izv. **37** (1991) 397-419.

[15] *A.M. Kamchatnov, M.V. Pavlov*, On generating functions in the AKNS hierarchy. Physics Letters A. **301** No. 3-4 (2002) 269-274.

[16] *O.I. Mokhov*, Lax pairs for compatible nonlocal Hamiltonian operators of hydrodynamic type. (Russian) Uspekhi Mat. Nauk **57**, No. 6 (2002) 189–190. *O.I. Mokhov*, Compatible metrics of constant Riemannian curvature: local geometry, nonlinear equations, and integrability. (Russian) Funktsional. Anal. i Prilozhen. **36** (2002), No. 3, 36–47, 96; translation in Funct. Anal. Appl. **36** (2002), No. 3, 196–204. *O.I. Mokhov*, A Lax pair for nonsingular bundles of metrics of constant Riemannian curvature. (Russian) Uspekhi Mat. Nauk **57** (2002), No. 3(345), 155–156; translation in Russian Math. Surveys **57** (2002), No. 3, 603–605.

[17] *M.A. Kamchatnov, R.A. Kraenkel*, On the relationship between a 2x2 matrix and second order scalar spectral problems. J. Phys. A: Math. Gen. **35** (2002) L13-L18.